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Atoms and Molecules on Riemannian Symmetric Spaces

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In this announcement we shall describe a relation between atoms and molecules on a non-compact Riemannian symmetric space G/K, and consider a multiplier operator on the atomic Hardy space. This is continuous a line of study in [7]. The details will appear elsewhere.

§1. Introduction. Before to state the aim, we shall recall some results on the theory of Hardy space $H^p(\mathbf{R})$ $(0<p\leq\infty)$ on one dimensional Euclidean space $\mathbf{R}$. The classical Hardy space is the space of analytic functions f on the upper half plane $\{(x,t);x\in\mathbf{R},\ t>0\}$ with finite H^p-norm:

$$\|f\|_{H^p}=\sup_{t>0}\left(\int_{-\infty}^{+\infty}|f(x,t)|^p dx\right)^{1/p}<\infty. \tag{1.1}$$

Moreover, taking the limit as $t\to+0$, this space is identified with the subspace of $\mathcal{S}'(\mathbf{R})$ consisting of boundary distributions f(x,0). In this definition the concept of "analytic functions" is necessary. However, new characterizations of $H^p(\mathbf{R})$ are recently obtained without using the concept of analytic functions. That is, D.L.Burkholder-R.F.Gundy-M.L.Silverstein and C.Fefferman-E.M.Stein showed that $H^p(\mathbf{R})$ is chracterized by the tangential maximal functions:

$$f^*(x)=\sup_{(y,t)\varepsilon\Gamma(x)}|f(y,t)|, \tag{1.2}$$

where $\Gamma(x)=\{(y,t);\ y\varepsilon\mathbf{R},\ t>0,\ |x-y|<t\}$. They obtained the following

Theorem A ([1],[5]). $c_p\|f\|_{H^p}\leq\|f^*\|_{L^p}\leq C_p\|f\|_{H^p}$.

Moreover, R.Coifman showed that $H^p(\mathbf{R})$ $(0<p\leq 1)$ can be characterized in terms of "atoms". Let (p,q,s) be a triplet such that $0<p\leq 1$, $1<q\leq\infty$ and $s\varepsilon\mathbf{N}$, $s\geq[1/p-1]$. Then a (p,q,s)-atom is a mesurable function on $\mathbf{R}$ such that the support is contained in an interval I and satisfies the following two conditions:

$$\begin{aligned} &\text{(i)} \quad \|f\|_q\leq|I|^{1/q-1/p} \\ &\text{(ii)} \quad \int_{\mathbf{R}} f(t)t^k dt=0 \quad (0\leq k\leq s). \end{aligned} \tag{1.3}$$

Then the atomic Hardy space $H^p_{q,s}(\mathbf{R})$ is the space consisting of distributions of the form

$$f=\sum_{i=1}^{\infty}\lambda_i f_i, \tag{1.4}$$

where f_i's are (p,q,s)-atoms and $\lambda_i\geq 0$, $\Sigma\lambda_i^p<\infty$. He obtained

Theorem B ([2]). $H^p(\mathbf{R})=H^p_{q,s}(\mathbf{R})$ and $c_p\|f\|^p_{H^p}\leq\rho^p_{q,s}(f)\leq C_p\|f\|^p_{H^p}$, where $\rho^p_{q,s}(f)$ is defined by the infimum of $\Sigma\lambda_i^p$ being taken over all decompositions (1.4).

Here let us define molecules corresponding to atoms. For a quartet (p,q,s,ε) such that (p,q,s) is as above and $\varepsilon>\max(s, 1/p-1)$, we put $a=1-1/p+\varepsilon$ and $b=1-1/q+\varepsilon$. Then a (p,q,s,ε)-

molecule centerd at x_0 is a function f on **R** such that f, $f|x|^b$ belong to $L^q(\mathbf{R})$ and satisfies the following two conditions:

$$\begin{aligned} &\text{(i)} \quad \|f\|_q^{a/b}\|f|x-x_0|^b\|_q^{1-a/b}=M(f) < \infty, \\ &\text{(ii)} \quad \int_{\mathbf{R}} f(x)x^k dx=0 \quad (0\le k\le s). \end{aligned} \tag{1.5}$$

Then M.H.Taibleson-G.Weiss showed the following

Theorem C ([10]).
(i) If f is a (p,q,s)-atom, then f is a (p,q,s,ε)-molecule for all $\varepsilon>0$ and $M(f)\le C$, where C is independent of the atom.
(ii) If f is a (p,q,s,ε)-molecule, then $f\varepsilon H^p_{q,s}(\mathbf{R})$ and $\rho^p_{q,s}(f)\le C'M(f)$, where C' is independent of the molecule.

By many people, these concepts: maximal functions, atoms and molecules on **R** were extended to $\mathbf{R}^n$ and moreover, to the general setting of spaces of homogeneous type (cf. [3],[6],[8]). But, our aim in this note is to extend these concepts to non-compact symmetric spaces G/K, which are not of homogeneous type. In §2, we shall give some notations about G, and in §3, define "radial maximal functions" and "atoms" on G/K and obtain a relation between them. In §4, we shall introduce "molecules" on G/K and obtain a theorem corresponding to Theorem C in **R**. Next we shall constract an atomic Hardy space by using the K-biinvariant, (p,q,s)-atoms on G centered at the unit element of G, and in §5, give a slightly simple characterization of this spce. In last §6, we shall consider convolution (or multiplier) operators on it.

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§2. Notations. Let G be a connected, real rank one semisimple Lie group with finite center, G=KAN an Iwasawa decomposition of G and $\underline{g}=\underline{k}+\underline{a}+\underline{n}$ the corresponding decomposition of the Lie algebra $\underline{g}$ of G. For any real vector space V let V_c and V^* denote the complexification and the dual space of V respectively. Let α be a reduced simple root of $(\underline{g}_c, \underline{a}_c)$ and H_0 the element of $\underline{a}$ such that $\alpha(H_0)=1$. In the following we identify A (resp. $\underline{a}_c^*$) with **R** by $a_t=\exp(tH_0)\leftrightarrow\alpha(\log(a_t))=t$ (resp. $\lambda \leftrightarrow \lambda(H_0)$) and moreover, by using the Cartan decomposition $G=KCL(A^+)K$ of G, we identify each K-biinvariant function f on G with the even function on **R** defined by, which we denote by the same letter, $f(t(x))=f(a_{t(x)})=f(x)$ for $x=k_1a_{t(x)}k_2\in KCL(A^+)K$. Let m_1 and m_2 denote the multiplicities of the root α and 2α respectively and put $\rho=(m_1+2m_2)/2$. Then for any K-biinvariant functions f on G with compact support its integral on G is given by the integral on $\mathbf{R}^+$ with weight $\Delta(t)=(\operatorname{sh}t)^{m_1}(\operatorname{sh}2t)^{m_2}$:

$$\int_G f(x)\,dx = \int_0^\infty f(t)\Delta(t)\,dt. \tag{2.1}$$

Let B(r,x) denote the open ball with radius r and centerd at x and |B(r,x)| the volume of it, i.e. $B(r,x)=\{y\in G;\ \sigma(x^{-1}y)<r\}$, where $\sigma(x)$ is the Riemannian distance between x and the unit element e of G, and $|B(r,x)|=\int_{B(r,x)}1dg=\int_0^r\Delta(t)\,dt$. For simplicity we put B(r)=B(r,e). Then the order of |B(r)| with respect to r is given by

$$B(r) = \begin{cases} O(e^{2\rho r}) & (r\to\infty) \\ O(r^{m_1+m_2+1}) & (r\to 0). \end{cases} \tag{2.2}$$

This property means that G is <u>not of homogeneous type</u> in the sence of [3].

§3. Maximal functions and atoms on G/K. First we shall define maximal functions on G/K (see [7,§3]). Let ϕ be a K-biinvariant function on G with finite L^1-norm. Then for a positive number $\varepsilon>0$, we put

$$\phi_\varepsilon(x)=\frac{1}{\varepsilon}\,\frac{\Delta(t(x)/\varepsilon)}{\Delta(t(x))}\;\phi(t(x)/\varepsilon). \tag{3.1}$$

Now for any locally integrable functions f on G/K, we define the radial maximal function $M_\phi f$ of f as follows.

$$M_\phi f(x)=\sup_{\varepsilon>0}|f*\phi_\varepsilon(x)|, \tag{3.2}$$

where $*$ is the convolution on G. Then the following theorem is valid (see [7, Theroem 3.3]).

Theorem 3.1. If there exist a constant C and a positive number $\delta>0$ such that $|\phi(x)|\leq Ce^{-2\rho\sigma(x)/\delta}$ $(x\in G)$, the operator M_ϕ is of type (L^p,L^p) $(1<p\leq\infty)$ and of weak type (L^1,L^1).

Next we shall define atoms on G/K (see [7,§4]). Let (p,q,s) be a triplet such that $0<p\leq 1$, $2(\alpha+1)/3<q\leq\infty$ and $s\in\mathbf{N}$, $s\geq[2(\alpha+1)/(1/p-1)]$, where α and β are defined by $m_1=2(\alpha-\beta)$ and $m_2=2\beta$. Then we say that a function f on G/K is a (p,q,s)-atom centered at x if the support is contained in an open ball B(r,x) and satisfies the following conditions:

$$\begin{array}{ll}\text{(i)} & \|f\|_q \leq |B(r)|^{1/q-1/p},\\ \text{(ii)} & \text{if } r<r_p=(\alpha+1)p/\rho(1-p),\ \text{then}\\ & \int_0^\infty f_{x,K}(t)\,t^k\Delta(t)\,dt=0\quad(0\leq k\leq s),\end{array} \tag{3.3}$$

where $f_{x,K}$ is the K-biinvariant function on G defined by $f_{x,K}(g)=$

$\int_K f(xkg)dk$. Of course, if we put $\alpha=\beta=-1/2$, i.e., $\Delta=1$ and $\rho=0$, this definition of atoms on G/K coincides with one on $\mathbf{R}$. If f satisfies the condition (ii) of (3.3), we say that f has <u>vanishing monents</u>. Here we define the <u>modified</u> radial maximal function $M'_\phi f$ for $f\in L^q(G/K)$ $(1\leq q\leq\infty)$ as follows.

$$M'_\phi f(x)= \sup_{0<\varepsilon<\varepsilon_p} |f*\phi_\varepsilon(x)|, \tag{3.4}$$

where $\varepsilon_p=(1-1/\delta)/(1-1/p)$ if $|\phi(t)|\leq Ce^{-2\rho|t|/\delta}$. Then the following theorem was obtained in [7,Theroem 4.1].

<u>Theroem 3.2</u>. <u>Let</u> $G\neq SL(2,\mathbf{R})$ <u>and</u> (p,q,s) <u>be as above</u>. <u>If there exist a constant</u> C, $0<\delta<1$ <u>and</u> $\lambda>1/p$ $(0<p<1)$ <u>such that</u> $|((\frac{d}{dt})^\ell\phi(t))(1+|t|)^\ell|\leq Ce^{-2\rho|t|/\delta}(1+|t|)^{-\lambda}$ <u>for all</u> $0\leq\ell\leq s+1$, <u>then there exists a constant</u> $c=c(C,p,q,s,\delta,\lambda)$ <u>such that</u> $\|M'_\phi f_{x,K}\|_p\leq c$ <u>for all</u> (p,q,s)-<u>atoms</u> f <u>on</u> G/K.

<u>§4. Molecules on G/K</u>. In the following we shall restrict our attension to K-biinvariant functions on G. Then the natural extension to G/K of the definition of molecules centered at 0 in $\mathbf{R}$ is given as follows. Let (p,q,s,ε) be a quartet such that (p,q,s) is as above , $\varepsilon>1/p-1$ and put $a=1-1/p+\varepsilon$, $b=1-1/q+\varepsilon$. Let B(x) denote the K-biinvariant function on G defined by $B(x)=|B(\sigma(x))|$. Then we say that a function f is a K-biinvariant, (p,q,s,ε)-<u>molecule centered at</u> e if it satisfies the following two conditions:

$$\begin{aligned} &\text{(i)} \quad \|f\|_q^{a/b}\|fB^b\|_q^{1-a/b}=M(f) < \infty, \\ &\text{(ii)} \quad \int_0^\infty f(t)t^k\Delta(t)dt=0 \ (0\leq k\leq s) \\ &\qquad \text{or } \|f\|_q \leq |B(r_p)|^{a-b}. \end{aligned} \tag{4.1}$$

Of course, if we put $\alpha=\beta=-1/2$, this definition coincides with one of (p,q,s,ε)-molecules centered at 0 in **R**. Then we can obtain the following

Theorem 4.1.

(i) *If* f *is a K-biinvariant*, (p,q,s)-*atom centered at* e, *then* f *is a K-biinvariant*, *molecule centered at* e *for all* $\varepsilon>0$ *and* $M(f)\leq C$, *where* C *is independent of the atom*.

(ii) *If* f *is a K-biinvariant*, (p,q,s,ε)-*molecule centered at* e *with vanishing moments*, *then* f *has an atomic decomposition* $f=\Sigma\lambda_i f_i$ *such that* f_i's *are K-biinvariant*, (p,q,s)-*atoms centered at* e *with vanishing moments and* $\lambda_i\geq 0$, $(\Sigma\lambda_i^p)^{1/p}\leq C'M(f)(1+N(f))^s$, *where* C' *is independent of the molecule and* N(f) *is defined by* $\|f\|_q=M(f)|B(N(f))|^{a-b}$.

Sketch of the proof: As in **R**, (i) is obvious from the definition. To prove (ii), without loss of generality, we may assume that $M(f)=1$. We define the number $N=N(f)$ by $\|f\|_q=|B(N)|^{a-b}$ and k_0 by the smallest integer such that $2^{k_0}N\geq 1$. Then we put

$$(4.2)\qquad G=\bigcup_{k=0}^{\infty} B_k,\quad B_k=\begin{cases} B(0,N) & (k=0)\\ B(2^{k-1}N,2^kN) & (0<k\leq k_0)\\ B(N_0+k-k_0-1,N_0+k-k_0) & (k_0<k),\end{cases}$$

where $B(r,r')=B(r)^c\cap B(r')$ and $N_0=2^{k_0}N$. Let f_k denote the restriction of f to B_k. Obviously, $f=\Sigma f_k$ and f_k's are K-biinvariant functions on G. To obtain the desired decomposition, we modify this to the desired one as in **R** (see [10,Theorem 2.9]). In this step we use the following lemma.

Lemma 4.2. For each k, there exist K-biinvariant functions h_k^i $(0\le i\le s)$ satisfying the following conditions:

(i) $\mathrm{supp}(h_k^i) \subset B_k \quad (0\le i\le s)$,

(ii) $\int_0^\infty h_k^i(t)t^j\Delta(t)dt=\delta_{ij} \quad (0\le i,j\le s)$,

(iii)

$$\| h_k^i\|_\infty \le C \begin{cases} N^{-(i+2\alpha+2)} & (k=0,\ N<1) \\ N^{s-i}|B(N)|^{-1} & (k=0,\ N\ge 1) \\ (2^{k-1}N)^{-(i+2\alpha+2)} & (0<k\le k_0) \\ N_0^{s-i}(k-k_0)^{s-i}|B(N_0+k-k_0+1)|^{-1} & (k_0<k). \end{cases}$$

Remark 1. If we use the decomposition of G such that $G=\bigcup_{k=0}^{\infty} B_k'$, $B_k'=B(2^{k-1}N,2^kN)$ instead of (4.2) (this corresponds to the case of $\mathbf{R}$), we have an atomic decomposition $f=\Sigma\lambda_i f_i$ such that $(\Sigma\lambda_i^p)^{1/p} \le C'M(f)e^{2\rho N(f)}$.

Remark 2. When f is a K-biinvariant (p,q,s,ε)-molecule $(0<p<1)$ which satisfies the latter condition of (ii) in (4.1), the similar result is valid. In this case f has an atomic decomposition consisting of atoms which satisfy (i) in (3.3) only.

§5. Atomic Hardy space on G/K. Let (p,q,s) be as above. Now let $L_+^p=L_+^p(G//K)$ denote the space of all K-biinvariant functions f on G having a non-increasing, K-biinvariant function $f^+\varepsilon L^p(G)$ such that $|f|\le f^+$. We call such a f^+ the L^p non-increasing dominator (L^p n.i.d.). In this section we shall consider the following three spaces:

$$°L_+^p=\{f\varepsilon L_+^p;\ f \text{ has a } L^p \text{ n.i.d. } f^+ \text{ such that } |B(r)|^{-1}\int_{B(r)_c} f(x)dx \le f^+(r)\}. \qquad (5.1)$$

$H^p=\{f\varepsilon L^1_{loc}(G//K);\ M'_\phi f\varepsilon L^p_+$ for all ϕ satisfying the condition in Theorem 3.2$\}$.

$H^p_{q,s}=\{f=\Sigma\lambda_i f_i$; all f_i's are K-biinvariant, (p,q,s)-atoms centered at e, $\lambda_i\geq 0$ and $\Sigma\lambda_i^p<\infty\}$.

Then we put $\rho^p_+(f)=\inf_{f^+}\|f^+\|^p_p$ for $f\varepsilon {}^\circ L^p_+$, where the infimum being taken over all L^p n.i.d. f^+ of f satisfying (5.1), $\rho^p(f)=\sup_\phi\inf\|(M'f)^+\|^p_p$ for $f\varepsilon H^p$, where the supremum (resp. the infimum) being taken over all ϕ satisfying the condition in Theorem 3.2. with C=1 (resp. all L^p n.i.d. of M'f) and $\rho^p_{q,s}(f)=\inf\Sigma\lambda_i^p$ for f $\varepsilon H^p_{q,s}$, where the infimum being taken over all K-biinvariant (p,q,s) -atomic decompositions of f. Obviously, $H^p_{q,s}\subset H^p_{q',s}$ $(q\geq q')$. The following proposition was obtained in [7,Proposition 5.1].

<u>Proposition 5.1</u>. $H^p_{q,s}\subset H^p$.

Moreover we can prove

<u>Theorem 5.2</u>. $H^p_{\infty,0}={}^\circ L^p_+$ <u>and</u> $\rho^p_{\infty,0}\sim\rho^p_+$.

Sketch of the proof: Let f be in $H^p_{\infty,0}$. Then f has an atomic decomposition $f=\Sigma\lambda_i f_i$ such that all f_i's are K-biinvariant, $(p,\infty,0)$ -atoms centered at e. That is, $\mathrm{supp}(f_i)\subset B(r_i)$ and $\|f_i\|_\infty\leq |B(r_i)|^{-1/p}$. Therefore, $|f|\leq\Sigma\lambda_i|f_i|\leq\sum_{\sigma(x)\leq r_i}\lambda_i|B(r_i)|^{-1/p}$. Here we define f^+ by the right hand side. Then we can show that f^+ is a L^p n.i.d. of f satisfying the condition (5.1). To prove the converse, we use Theorem 4.1 (ii) and the similar argument in the proof of the theorem.

<u>Corollary 5.3</u>. $H^p_{\infty,0}$ <u>is complete</u>.

Conjecture. $H^p_{\infty,s} = {}^\circ L^p_+ = H^p$.

Remark. As in $\mathbf{R}$, if the integral: $|B(r)|^{-1} \int_{B(r)_c} f(x)\,dx$ can be expressed suitably in terms of the convolutions on G and be bounded by the maximal functions of f, this conjecture is valid.

§6. Multiplier operators on $H^p_{q,s}$. In this section we shall consider convolution (or multiplier) operators on $H^p_{q,s}$. First, as in $\mathbf{R}$, we see that

Proposition 6.1. *If a linear operator* T *maps each K-biinvariant, (p,q,s)-atoms centered at* e *into a K-biinvariant, (p,q,s)-molecule* T(f) *centered at* e *and* $M(T(f)) \leq C$, *where* C *is independent of the atom* f, *then* T *is a bounded operator on* $H^p_{q,s}$.

By using this proposition we can obtain the following results. For a K-biinvariant function f on G (resp. an even function μ on $\underline{a}^*$) with a suitable condition, the Spherical Fourier transform $\hat{f}$ of f (resp. the inverse Fourier transform $\check{\mu}$ of μ) is defined as follows (cf. [11,Chap.9.2]).

$$\hat{f}(\nu) = \int_G f(x)\phi_\nu(x)\,dx$$

$$(\text{resp. } \check{\mu}(x) = \int_{\underline{a}^*} \mu(\nu)\phi_\nu(x)|C(\nu)|^{-2}d\nu).$$

Now we put $F(\xi) = \{\nu \in \underline{a}^*_c;\ |\mathrm{Im}(\xi)| < \xi\rho\}$. Then we have the following

Theorem 6.1. *Suppose that* μ *is an even function on* $\underline{a}^*$ *such that* μ *is bounded and holomorphic on* $F(\xi)$ $(\xi > 2/p-1)$ *and* $\mu(\nu)(1+|\nu|)^{1-[p]} C(-\nu)^{-1} \in L^1(\mathbf{R}+\sqrt{-1}\xi\rho)$. *Then if the multiplier operator* T_μ, i.e., $T_\mu(f) = (\mu\hat{f})^\vee$, *is of type* (L^∞, L^∞), T_μ *is also of type* $(H^p_{\infty,0}, H^p_{\infty,0})$ *for* $2\alpha+2/2\alpha+3 \leq p \leq 1$.

Moreover, using this theorem and Corollary 5.3, we can obtain

Corollary 6.3. *Suppose that* m *is a K-biinvariant function on* G *with finite* L^1-*norm and* $\hat{m}(\nu)\ C(-\nu)^{-1} \in L^1(\mathbf{R}+\sqrt{-1}\rho)$. *Then the convolution operator* T_m, i.e., $T_m(f)=m*f$, *is of type* $(H^1_{\infty,0}, H^1_{\infty,0})$.

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